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DSM通信に対するFDMA最適性とそれに基づいた解法

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1 Introduction

The digital subscriber line (DSL) is widely used for the device of the broadband Internet access. Since the DSL system transports the digital data through the existing telephone cables, the installation cost is rather low. On the other hand, the crosstalk, which is the inter-user electromagnetic interference due to the structure of the telephone cable with common bundled copper wires, is the dominant source of the data loss in DSL system.

Dynamic spectrum management (DSM) is one of effective techniques to mitigate such a loss, in which the center office controls the powers allocated to each user so that the crosstalk is cancelled out and the total system throughput improves. Recently, many researchers study the possible application of DSM techniques to not only DSL but also the wireless network system, etc. One of the most popular measure for evaluating the total throughput is sum rate (the sum of all users' data rates). However, the sum rate function is non-concave for allocated powers, and hence, the problem of maximizing the sum rate may have many local maxima and usual optimization techniques are not suitable.

Many researchers have tried to employ algorithms for solving sum rate maximization problem. Cherubini et al. proposed a simulated annealing method [5], but its convergence is rather slow. Recently, some researchers proposed optimal spectrum balancing (OSB) algorithms based on duality theory [1, 4, 9, 13]. In these algorithms, the authors aim to solve the dual problem instead of the original sum rate maximization problem. Although the dual problem can be decomposed to smaller-dimensional problems and their objective functions are convex, it is difficult to solve it exactly since the evaluation of the dual objective function involves a non-concave maximization. Moreover, the duality gap may still exist when the crosstalk is strong.

Based on the game theoretical ideas, several kinds of water-filling algorithms were proposed [3, 6, 10–12]. Among them, the most popular one is the iterative water-filling algorithm (IWFA), which maximizes a certain user's data rate by the water-filling procedure in each step, and by repeating the water-filling steps successively, one can finally obtain the power allocation which may give rise to a good sum rate. In actual, the obtained solution is the Nash equilibrium where each player's payoff function is correspondent to his/her data rate. If the crosstalk is small enough, then the uniqueness of Nash equilibrium is guaranteed [11] and IWFA finds the unique Nash point efficiently. However, when the crosstalk is strong, obtained solution by IWFA is not good enough, and it often fails to converge to the Nash equilibrium.

Main purpose of the paper is to propose an algorithm which is efficient for sum rate maximization problem with strong crosstalk, and to give a theoretical background which justify the efficiency of our algorithm. The key concept is FDMA (Frequency Division Multiple Access), which means that the power is allocated exclusively to one user for each tone. In the modern DSL system, the discrete multitone (DMT) modulation is employed, in which the feasible frequency band is divided to several tones so that the data stream is carried parallelly according to the powers allocated to each tone. The number of tone is 256 for Asymmetric DSL (ADSL) and 4096 for Very high bit rate DSL (VDSL). Usually, the number of tone is much greater than that of users.

In this paper, we show that, if the crosstalk coefficients are larger than certain values (approximately 1/2), then the optimal solution of sum rate maximization problem becomes FDMA.

One may think it is trivial since the crosstalk interference is active only when different users' powers coexist in the same tone. However, it is not easy to estimate the crosstalk criterion whether the optimal power allocation is FDMA or not. In actual, if the crosstalk is small enough, then a widespread and non-tone-monopolistic power allocation tends to be better than tone-monopolistic power allocation, since each user's achievable data rate is concave for his/her own power vector.

We also propose some algorithms based on the greedy method and the dual decomposition, which give a sufficiently large sum rate under assumption that the optimal solution is FDMA. The numerical results indicate that our algorithms calculate the solution in a sufficiently short time, and show better performance than IWFA when the crosstalk coefficients and/or the power budgets are large enough.

This paper is organized as follows. In Section 2, we describe the system model and give some mathematical preliminaries. In Section 3, we derive a sufficient condition under which the global optimum of the sum-rate maximization problem possesses the FDMA structure. In Section 4, we further provide a sufficient condition for the existence of a local maxima of the sum-rate function (subject to individual power constraints) that has the FDMA structure. In Section 5, we establish the NP-hardness of the sum-rate maximization problem and propose a simple distributed algorithm and two polynomial time combinatorial search algorithms for finding a FDMA solution with maximal sum-rate. Numerical results are reported in Section 6, and the concluding remarks are given in Section 7.

Throughout the paper, we use the following notations. We denote the set of frequency tones and users by $\mathcal{N}$ and $\mathcal{K}$, respectively, i.e., $\mathcal{N} := \{1, \ldots, N\}$ and $\mathcal{K} := \{1, \ldots, K\}$. Also, we use superscript n to denote the frequency tone index and subscript k to denote the user index.

2 Channel model and sum-rate maximization

We first describe the frequency selective Gaussian interference channel model and the mathematical formulation of the sum-rate maximization problem.

Suppose that there are K users sharing a common spectrum which is divided into N frequency tones numbered by $\{1, 2, ..., N\}$. Let $S_k^n \geq 0$ be the transmission power for user k at tone n. Then, assuming that the interference is treated as white noise, we can write user k's achievable data rate R_k^n at tone n [2] as

$$R_k^n(S_1^n, \ldots, S_K^n) := \log\left(1 + \frac{S_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n S_l^n}\right), \tag{2.1}$$

where σ_k^n denotes the (normalized) background noise power, and α_{lk}^n is the (normalized) crosstalk coefficient from user l to user k at tone n. Due to normalization, we have $\alpha_{kk}^n = 1$ for all k.

Throughout, we assume that transmitter k's power is bounded by the power budget $P_k > 0$, i.e.,

$$\sum_{n=1}^{N} S_k^n \leq P_k, \qquad \text{for } k \in \mathcal{K}.$$

For a given power allocation $\{S_k^n\}$, transmitter k's total achievable data rate is given by $\sum_{n=1}^N R_k^n$ and the total sum-rate is given by $\sum_{k=1}^K \sum_{n=1}^N R_k^n$. Hence, the sum-rate maximization problem can be written as follows:

$$\begin{array}{ll} \underset{\{S_1^n, \ldots, S_K^n\}_{n=1}^N}{\text{maximize}} & \displaystyle\sum_{k=1}^{K} \sum_{n=1}^{N} \log\left(1 + \frac{S_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n S_l^n}\right) \\ \text{subject to} & \displaystyle\sum_{n=1}^{N} S_k^n \leq P_k, \quad S_k^n \geq 0 \quad n \in \mathcal{N},\ k \in \mathcal{K}. \end{array} \tag{2.2}$$

It can be easily seen that user k's total achievable data rate $\sum_{n=1}^{N} R_k^n$ is concave for user k's power vector $(S_k^1, \ldots, S_k^N)$ when other users' power vectors are fixed. However, the total sum-rate function $\sum_{k=1}^{K}\sum_{n=1}^{N} R_k^n$ is in general non-concave even if other users' powers are fixed, since user k's power S_k^n appears in the denominators of other users' data rate function.

When interference is absent (or small), it can be easily checked [7] that signal spreading across spectrum is optimal. In other words, if the crosstalk coefficients are sufficiently small, then all frequency tones should be utilized by all users. On the other hand, if the crosstalk coefficients are large, then the communication system becomes interference limited, and spectrum sharing is no longer optimal. Intuitively, FDMA should yield a larger sum-rate in this case. Mathematically, FDMA property is defined as follows:

Definition 2.1 *A feasible solution* $\{S_1^n, \ldots, S_K^n\}_{n=1}^N$ *of the sum-rate maximization problem* (2.2) *is said to have FDMA property, if the following implication holds for all* $(n,k) \in \mathcal{N} \times \mathcal{K}$:

$$S_k^n > 0 \quad \Longrightarrow \quad S_l^n = 0, \qquad \forall\, l \neq k.$$

To simplify our notations, we let $\mathbf{S}^n$, $\mathbf{S}_k$, and $\mathbf{S}$ denote the power vectors at tone n, for user k, and in the whole system, respectively, i.e.,

$$\mathbf{S}^n := (S_1^n, \ldots, S_K^n) \in \Re^K, \quad \mathbf{S}_k := (S_k^1, \ldots, S_k^n) \in \Re^N, \quad \text{and} \quad \mathbf{S} := (S_1^1, \ldots, S_K^N) \in \Re^{NK}.$$

We denote the power budget vector by $\mathbf{P}$, i.e., $\mathbf{P} := (P_1, \ldots, P_K) \in \Re^K$. Also, we denote the noise plus interference power for user k at tone n, and the sum of all users' data rates at tone n by X_k^n and f^n, respectively, i.e.,

$$X_k^n(\mathbf{S}^n) := \sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n S_l^n, \qquad f^n(\mathbf{S}^n) := \sum_{k=1}^{K} R_k^n(\mathbf{S}^n) = \sum_{k=1}^{K} \log\left(1 + \frac{S_k^n}{X_k^n}\right). \tag{2.3}$$

Note that X_k^n and f^n depend on $\mathbf{S}^n$ only. The following index sets which will be convenient for describing the FDMA property of a feasible power vector.

Definition 2.2 *For a feasible solution* $\mathbf{S}$ *of problem* (2.2), *we define the following sets.*

$$\begin{aligned} \mathcal{T}(\mathbf{S}) &:= \{(n,k) \mid S_k^n > 0\} \subseteq \mathcal{N} \times \mathcal{K}, \\ \mathcal{T}_k(\mathbf{S}_k) &:= \{n \mid S_k^n > 0\} \subseteq \mathcal{N}, \\ \mathcal{T}^n(\mathbf{S}^n) &:= \{k \mid S_k^n > 0\} \subseteq \mathcal{K}. \end{aligned}$$

Note that $\mathcal{T}_k(\mathbf{S}_k)$ denotes the set of all tones used by user k, and $\mathcal{T}^n(\mathbf{S}^n)$ denotes the set of all users using tone n.

3 Sum-rate optimality of FDMA

As mentioned earlier, we expect that an FDMA-type power allocation will maximize the sum-rate when the crosstalk coefficients are sufficiently large. In this section we show the validity of this claim and derive an explicit bound on the crosstalk coefficients which will ensure the existence of an optimal FDMA type solution. For more detailed theories and proofs, see [8].

Let us first introduce a condition on a feasible power allocation vector $\mathbf{S}$.

Condition 1 *There holds*

(a) $\min_{k \in \mathcal{K}} |\mathcal{T}_k(\mathbf{S}_k)| \geq C$, *for some integer* $C \geq 2$.

(b) $\sum_{n=1}^{N} \mathbf{S}^n = \mathbf{P}$.

In other words, every user uses at least two tones and exhausts his power budget. Moreover, we assume that the global maximum satisfies this condition.

Assumption A *Any global maximum of problem* (2.2) *satisfies Condition* 1 *for some* $C \geq 2$.

Assumption A is difficult to verify since the global maximum is not known a priori. However, in a practical DSL system, the number of tones N is usually much larger than the number of users K, i.e., $K \ll N$, and the power budget for each user is sufficiently high. In such cases, Condition 1(a) is supposed to be satisfied with a large C. The following proposition shows that, when all the crosstalk coefficients are greater than or equal to 1/2, the lower bound C of $\min_{k\in\mathcal{K}} |\mathcal{T}_k(\mathbf{S}_k)|$ can be evaluated by using the constants in the problem only.

Proposition 3.1 *Suppose* $\alpha_{lk}^n \geq \frac{1}{2}$ *for all* $l, k \in \mathcal{K}$ *and* $n \in \mathcal{N}$. *Let* $C \in [2, N]$ *be an arbitrary integer. If there exists another integer* $m \in [C, N]$ *such that*

$$1 + \frac{\rho_0}{m} > \left(1 + \frac{\rho_M}{C-1}\right)^{\frac{C-1}{m}} \left(1 + \frac{K\rho_a}{N-m+1}\right), \tag{3.1}$$

where

$$\rho_0 := \min_{(n,k)\in\mathcal{N}\times\mathcal{K}} \frac{P_k}{\sigma_k^n}, \quad \rho_M := \max_{(n,k)\in\mathcal{N}\times\mathcal{K}} \frac{P_k}{\sigma_k^n}, \quad \rho_a := \frac{\frac{1}{K}\sum_{k=1}^K P_k}{\min_{(n,k)\in\mathcal{N}\times\mathcal{K}} \sigma_k^n},$$

then $\min_{k\in\mathcal{K}} |\mathcal{T}_k(\mathbf{S}_k)| \geq C$ *for any global maximizer* $\mathbf{S}$ *of the sum-rate maximization problem* (2.2).

We now show that, if Assumption A holds and the normalized crosstalk coefficients are sufficiently greater than 1/2, then optimal spectrum sharing strategy must be FDMA.

Theorem 3.1 *Suppose that Assumption* A *holds. Then, any global maximum of problem* (2.2) *must be FDMA, provided that*

$$\alpha_{lk}^n > \frac{1}{2} \quad \text{and} \quad \alpha_{lk}^n \alpha_{kl}^n > \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^2$$

for all $n \in \mathcal{N}$ *and* $(k, l) \in \mathcal{K} \times \mathcal{K}$ *with* $k \neq l$.

When C is sufficiently large, say, $C > 100$, we have $1 + \frac{1}{C-1} \approx 1$. In this case, the condition $\alpha_{lk}^n \alpha_{kl}^n > \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^2$ is essentially implied by the condition $\alpha_{lk}^n > \frac{1}{2}$. Thus, Theorem 3.1 shows that if the normalized crosstalk coefficients are sufficiently greater than 1/2, then the optimal spectrum sharing strategy must be FDMA.

Next, we restrict ourselves on the two-user case ($K = 2$) and show the optimality of FDMA strategy under a weaker condition than that of Theorem 3.1. Specifically, we show that the condition $\min\{\alpha_{12}^n, \alpha_{21}^n\} > \frac{1}{2}$ can be dropped when $K = 2$, and the optimality of FDMA strategies is ensured under the condition $\alpha_{21}^n \alpha_{12}^n > \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^2$ alone.

Theorem 3.2 *Suppose that* $K = 2$ *and Assumption A holds for some* $C \geq 2$. *If*

$$\alpha_{12}^n \alpha_{21}^n > \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^2$$

for all $n \in \mathcal{N}$, *then the global maximum of sum-rate maximization problem* (2.2) *is FDMA.*

Before closing this section, we provide a proposition where the condition $\alpha_{lk}^n \geq 1/2$ in Proposition 3.1 is replaced by $\alpha_{12}^n \alpha_{21}^n > 1/4$. Although the proposition gives a lower bound C for $\min_{k \in \mathcal{K}} |\mathcal{T}_k(\mathbf{S}_k)|$, it may not be sufficiently tight.

Proposition 3.2 *Suppose that $K = 2$ and $\alpha_{12}^n \alpha_{21}^n > 1/4$ for all $n \in \mathcal{N}$. Let $C \in [2, N]$ be an arbitrary integer. If there exists another integer $m \in [C, N]$ such that (3.1) holds, then $\min_{k \in \mathcal{K}} |\mathcal{T}_k(\mathbf{S}_k)| \geq C$ for any global maximizer $\mathbf{S}$ of problem (2.2).*

4 Existence of a locally optimal FDMA solution

The goal of this section is to derive some weaker sufficient conditions which will guarantee the existence of a FDMA type *local* maxima. Although these conditions do not guarantee the global optimum to be FDMA, the numerical results in Section 6 show that, under such conditions, FDMA type power allocations often show better performance than the solutions obtained by the iterative water-filling algorithm (IWFA).

Let us define the set of FDMA type frequency allocations by

$$\mathcal{FDM} := \left\{ \mathcal{L} \;\middle|\; \min_{k \in \mathcal{K}} |\mathcal{L}_k| \geq 1, \ \cup_{k=1}^{K} \mathcal{L}_k = \mathcal{N}, \ \text{and} \ \mathcal{L}_k \cap \mathcal{L}_l = \emptyset \quad (\forall k \neq l) \right\}.$$

Here $\mathcal{L}_k$ represents the set of frequency tones allocated to user k. For any $\mathcal{L} \in \mathcal{FDM}$, we consider the following $\mathcal{L}$-restricted sum-rate maximization problem (denoted by $\mathbf{SRMP}(\mathcal{L})$):

$$\mathbf{SRMP}(\mathcal{L}) \qquad \begin{array}{ll} \underset{\{S_k^1,\dots,S_k^N\}_{k=1}^K}{\text{maximize}} & \displaystyle\sum_{n=1}^{N} f^n(\mathbf{S}^n) = \sum_{k=1}^{K} \sum_{n \in \mathcal{L}_k} \log\left(1 + \frac{S_k^n}{\sigma_k^n}\right) \\ \text{subject to} & \displaystyle\sum_{n=1}^{N} \mathbf{S}^n \leq \mathbf{P}, \ \ S_k^n \geq 0 \ (n \in \mathcal{L}_k), \ \ S_k^n = 0 \ (n \notin \mathcal{L}_k), \ \ k \in \mathcal{K}, \end{array}$$

where the equality for the objective function (sum-rate function) is valid since FDMA requirement implies that there is no interference among users. Notice that $\mathbf{SRMP}(\mathcal{L})$ is a concave maximization, and does not involve any crosstalk coefficient α_{lk}^n. Moreover, $\mathbf{SRMP}(\mathcal{L})$ is completely separable with respect to each user k, implying that $\mathbf{SRMP}(\mathcal{L})$ can be decomposed into the following K independent rate maximization problems:

$$\mathbf{RMP}(\mathcal{L}_k) \qquad \begin{array}{ll} \underset{\{S_k^1,\dots,S_k^N\}}{\text{maximize}} & \displaystyle\sum_{n \in \mathcal{L}_k} \log\left(1 + \frac{S_k^n}{\sigma_k^n}\right) \\ \text{subject to} & \displaystyle\sum_{n \in \mathcal{L}_k} S_k^n \leq P_k, \ \ S_k^n \geq 0 \ (n \in \mathcal{L}_k), \ \ S_k^n = 0 \ (n \notin \mathcal{L}_k), \end{array}$$

It is known that each user's rate maximization problem $\mathbf{RMP}(\mathcal{L}_k)$ can be solved by the water-filling procedure [3,6,10–12]. To focus our analysis on the interference in the system, we make the following high signal to noise ratio assumption.

Assumption B *For all $k \in \mathcal{K}$, there holds*

$$\gamma_k := \frac{P_k + \sum_{n \in \mathcal{L}_k} \sigma_k^n}{|\mathcal{L}_k|} > \max_{n \in \mathcal{L}_k} \sigma_k^n. \tag{4.1}$$

Under Assumption B, the water level (see [3,6,10–12]) is equal to γ_k, and the global maximum of $\mathbf{SRMP}(\mathcal{L})$ can be described explicitly.

$$\begin{cases} S_k^n = \gamma_k - \sigma_k^n & (\forall n \in \mathcal{L}_k) \\ S_k^n = 0 & (\forall n \notin \mathcal{L}_k) \end{cases} \tag{4.2}$$

for $k = 1, \ldots, K$.

The following proposition gives sufficient conditions under which problem (2.2) has a FDMA local maximum.

Proposition 4.1 *Let the tone allocation set be given by $\mathcal{L} \in \mathcal{FDM}$, and γ_k be defined by* (4.1). *Suppose that Assumption* B *holds. If*

$$\frac{1}{\sigma_k^n + \alpha_{lk}^n(\gamma_l - \sigma_l^n)} - \alpha_{kl}^n\left(\frac{1}{\sigma_l^n} - \frac{1}{\gamma_l}\right) \leq \frac{1}{\gamma_k} \tag{4.3}$$

for any $(k,l) \in \mathcal{K} \times \mathcal{K}$ with $k \neq l$ and $n \in \mathcal{L}_l$, then the global maximum (4.2) *of* **SRMP**$(\mathcal{L})$ *is a local maximum of sum-rate maximization problem* (2.2).

Although the condition (4.3) can be verified beforehand, it involves all possible combinations for k, l and n, and concerns only a given tone allocation $\mathcal{L}$. This makes it inconvenient to apply Proposition 4.1 in practice. In the following corollary result, we simplify the conditions of Proposition 4.1 so as to improve its applicability in practice.

Theorem 4.1 *Let C be an arbitrary integer such that $1 \leq C \leq N/K$, and denote $P_M := \max_k P_k$, $P_0 := \min_k P_k$, $\sigma_M := \max_{n,k} \sigma_k^n$, $\sigma_0 := \min_{n,k} \sigma_k^n$, $\alpha_0 := \min_{n,k,l\,(k\neq l)} \alpha_{lk}^n$, $A_0 := \min_{n,k,l\,(k\neq l)} \alpha_{lk}^n \alpha_{kl}^n$, $\gamma_M := P_M/C + \sigma_M$, $\gamma_0 := P_0/(N-(K-1)C) + \sigma_0$. Suppose that the following inequalities hold:*

$$\gamma_0 > \sigma_M, \tag{4.4}$$

$$A_0\gamma_M(\gamma_0 - \sigma_M)^2 + \alpha_0(\gamma_M\sigma_0 + \gamma_0\sigma_M)(\gamma_0 - \sigma_M) \geq \sigma_M\gamma_0(\gamma_M - \sigma_0). \tag{4.5}$$

Then, for any tone set $\mathcal{L} \in \mathcal{FDM}$ such that $\min_{k\in\mathcal{K}} |\mathcal{L}_k| \geq C$, the global maximum of **SRMP**$(\mathcal{L})$ *is a local maximum of sum-rate maximization problem* (2.2). *Moreover, if*

$$P_0 \geq \Big(N - (K-1)C\Big)\left(\frac{1}{A_0} + \frac{1}{\sqrt{A_0}} + 1\right)\sigma_M, \tag{4.6}$$

then (4.5) *holds.*

Although condition (4.6) is more restrictive than (4.5), it is more intuitive and easier to apply in practice. Compared to our earlier results (Theorems 3.1 and 3.2), Theorem 4.1 shows the existence of a FDMA type local maxima for the sum-rate maximization problem (2.2) even when the crosstalk coefficients are small (but positive), so long as users' power budgets are sufficiently large.

5 Finding an optimal FDMA bandwidth allocation

In this section, we focus our attention on the more practical issue of how to design an optimal FDMA scheme for a multiuser communication system. The latter entails allocating the available set of frequency tones to the users in the system. Let us denote the set of FDMA solutions by

$$\mathcal{S} = \{\mathbf{S} \geq 0 \mid S_k^n S_l^n = 0,\ \forall\, k \neq l,\ \forall\, n\},$$

where the condition $S_k^n S_l^n = 0$ signifies that no frequency tone can be shared by any two users. Then, the optimal FDMA frequency allocation problem can be described as follows:

$$\begin{aligned} &\underset{\mathbf{S}}{\text{maximize}} && \sum_{k=1}^{K}\sum_{n=1}^{N} \log\left(1 + \frac{S_k^n}{\sigma_k^n}\right) \\ &\text{subject to} && \mathbf{S} \in \mathcal{S}, \quad \sum_{n=1}^{N} S_k^n \leq P_k, \quad k = 1, \ldots, K. \end{aligned} \tag{5.1}$$

where $\mathbf{S}$ denotes the (NK)-dimensional vector with entries equal to S_k^n. Notice that, due to the FDMA condition, the interference term $\sum_{l\neq k} \alpha_{lk}^n S_l^n$ is absent from the sum-rate objective function. This makes the objective function concave. However, problem (5.1) remains a nonconvex problem due to the nonconvex constraint $\mathbf{S} \in \mathcal{S}$. The following result shows that the optimization problem (5.1) is NP-hard, even in the case of two users.

Theorem 5.1 *For $K = 2$, the optimal bandwidth allocation problem* (5.1) *is NP-hard. Thus, the general sum-rate maximization problem* (2.2) *is also NP-hard, even in the two-user case.*

Given this negative result, we are naturally led to the problem of designing efficient polynomial time algorithms which can approximately maximize the sum-rates. In what follows, we propose three simple algorithms for computing an approximately optimal FDMA bandwidth allocations. (Here, we just describe the concrete algorithms and their backgrounds. For more detailed discussion, refer to [8].)

The first one is based on dual decomposition which tries to minimize Lagrange dual function subject to the Lagrange multiplier $\boldsymbol{\lambda} \geq 0$.

Algorithm 1 (Dual decomposition method)

Step 0 *Choose an initial point $\boldsymbol{\lambda}^{(0)} \geq 0$ and a stepsize $\alpha^{(0)} > 0$. Set $\nu = 0$.*

Step 1 *For all $(n,k) \in \mathcal{N} \times \mathcal{K}$, compute*

$$(\overline{S}_k^n)^{(\nu)} := \begin{cases} \mathcal{P}_k\left((\lambda_k^{(\nu)})^{-1} - \sigma_k^n\right) & \text{if } \lambda_k^{(\nu)} > 0 \\ P_k & \text{if } \lambda_k^{(\nu)} = 0, \end{cases}$$

$$(M_k^n)^{(\nu)} := \log\left(1 + \frac{(\overline{S}_k^n)^{(\nu)}}{\sigma_k^n}\right) - \lambda_k (\overline{S}_k^n)^{(\nu)},$$

where $\mathcal{P}(\cdot)$ denotes the projection of a real number onto the interval $[0, P_k]$. Moreover, for each $k = 1, \ldots, K$, set the FDMA tone assignment according to

$$\mathcal{N}_k(\boldsymbol{\lambda}^{(\nu)}) := \left\{ n \in \mathcal{N} \,\middle|\, (M_k^n)^{(\nu)} = \max_{k'=1,\ldots,K} (M_{k'}^n)^{(\nu)} \right\},$$

and calculate the subgradient by

$$g_k^{(\nu)} := P_k - \sum_{n \in \mathcal{N}_k(\boldsymbol{\lambda}^{(\nu)})} (\overline{S}_k^n)^{(\nu)}.$$

Step 2 *Update $\boldsymbol{\lambda}^{(\nu)}$ according to*

$$\lambda_k^{(\nu+1)} = \left[\lambda_k^{(\nu)} - \alpha^{(\nu)} g_k^{(\nu)}\right]_+, \quad k = 1, 2, \ldots, K,$$

where $[\cdot]_+$ denotes the positive part of a real number, and $\alpha^{(\nu)}$ is the stepsize calculated by an appropriate rule.

Step 3 *Go to Step 4 if the termination criterion is satisfied. Otherwise, set $\nu := \nu + 1$, and return to Step 1.*

Step 4 *If $\overline{\mathbf{S}}^{(\nu)}$ is feasible for problem* (5.1)*, then output it as the solution. Otherwise, choose $\overline{\nu}$ such that $\|g^{(\overline{\nu})}\| = \min\{\|g^{(0)}\|, \ldots, \|g^{(\nu)}\|\}$, and calculate the optimal power allocation $\mathbf{S}$ based on $\mathcal{N}_1(\boldsymbol{\lambda}^{(\overline{\nu})}), \ldots, \mathcal{N}_K(\boldsymbol{\lambda}^{(\overline{\nu})})$. Then, output $\mathbf{S}$ as the solution.*

We next present an efficient combinatorial greedy local search algorithm which has an overall complexity of $O(NK)$. In this algorithm, we fix the order of tones a priori, and then sequentially allocate each tone to the user who offers the largest rate increment. This algorithm can be written as follows.

Algorithm 2 (Local search algorithm A)

Step 0 *Permute the tones $n_1, \ldots, n_N$ arbitrarily so that $\{n_1, \ldots, n_N\} = \mathcal{N}$. Let $\mathcal{L}_k^{(0)} = \emptyset$ and $\overline{R}_k^{(0)} := 0$ for each $k = 1, \ldots, K$. Set $\nu := 0$.*

Step 1 *For each $k = 1, \ldots, K$, solve* **RMP**$(\mathcal{L}_k^{(\nu)} \cup \{n_{\nu+1}\})$ *and obtain its optimal value $\overline{R}'_k$.*

Step 2 *Find a $\overline{k}$ such that*

$$\overline{k} = \operatorname*{argmax}_{k \in \mathcal{K}} \left(\overline{R}'_k - \overline{R}_k^{(\nu)} \right).$$

Then, define $\mathcal{L}_k^{(\nu+1)}$ and $\overline{R}_k^{(\nu+1)}$ by

$$\mathcal{L}_k^{(\nu+1)} := \begin{cases} \mathcal{L}_k^{(\nu)} \cup \{n_{\nu+1}\} & (k = \overline{k}) \\ \mathcal{L}_k^{(\nu)} & (k \neq \overline{k}) \end{cases} \quad \text{and} \quad \overline{R}_k^{(\nu+1)} := \begin{cases} \overline{R}'_k & (k = \overline{k}) \\ \overline{R}_k^{(\nu)} & (k \neq \overline{k}) \end{cases}$$

for each $k = 1, \ldots, K$.

Step 3 *Set $\nu := \nu + 1$. If $\nu = N$, then terminate. Otherwise, return to Step 1.*

In Step 1, $\overline{R}'_k$ can be obtained by the water-filling procedure. In general, the obtained solution and sum-rate depend on the initial ordering of $\{n_1, \ldots, n_N\}$.

In Algorithm 2, we have fixed the order of tones beforehand, and then allocate a tone $n_{\nu+1}$ at the ν-th iteration. However, it is expected that the sum-rate will be improved by considering all the possible combinations of tones and users at each iteration. A direct implementation of such a procedure will result in a computational complexity of $O(N^2K)$. However, by sorting the noise parameters $\{\sigma_k^n\}$ appropriately, we can reduce its complexity to $O(NK \log N)$. We describe the algorithm in the following, where $\mathcal{L}_k^{(\nu)}$, $\overline{R}_k^{(\nu)}$, and $\overline{\mathcal{N}}^{(\nu)}$ denote user k's allocated tone set, user k's temporary data rate, and unallocated tone set at the ν-th iteration.

Algorithm 3 (Local search algorithm B)

Step 0 *For each $k = 1, \ldots, K$, sort the tone indices $\{n_1(k), \ldots, n_N(k)\} = \mathcal{N}$ so that*

$$\sigma_k^{n_1(k)} \leq \cdots \leq \sigma_k^{n_N(k)}.$$

Let $\mathcal{L}_k^{(0)} = \emptyset$, $R_k^{(0)} := 0$, and $\overline{\mathcal{N}}^{(0)} = \mathcal{N}$ for all $k \in \mathcal{K}$. Set $\nu := 0$.

Step 1 *For every $k = 1, \ldots, K$, perform the following steps:*

Step 1-1 *Find a tone $\overline{n}(k) := n_{i_-}(k)$ such that*

$$i_- = \min\left\{ i \;\middle|\; n_i(k) \in \overline{\mathcal{N}}^{(\nu)} \right\}.$$

Step 1-2 *Solve* **RMP**$(\mathcal{L}_k^{(\nu)} \cup \{\overline{n}(k)\})$ *and obtain its optimal value $\overline{R}'_k$.*

Step 2 *Find a $\overline{k} \in \mathcal{K}$ such that*

$$\overline{k} = \operatorname*{argmax}_{k \in \mathcal{K}} \left(\overline{R}'_k - \overline{R}^{(\nu)}_k \right).$$

Define $\mathcal{L}^{(\nu+1)}_k$ and $\overline{R}^{(\nu+1)}_k$ by

$$\mathcal{L}^{(\nu+1)}_k := \begin{cases} \mathcal{L}^{(\nu)}_k \cup \{\overline{n}(k)\} & (k = \overline{k}) \\ \mathcal{L}^{(\nu)}_k & (k \neq \overline{k}) \end{cases} \quad \text{and} \quad \overline{R}^{(\nu+1)}_k := \begin{cases} \overline{R}'_k & (k = \overline{k}) \\ \overline{R}^{(\nu)}_k & (k \neq \overline{k}) \end{cases}$$

for each $k = 1, \dots, K$. Then, let $\overline{\mathcal{N}}^{(\nu+1)} := \overline{\mathcal{N}}^{(\nu)} \setminus \{\overline{n}(\overline{k})\}$.

Step 3 *If $\overline{\mathcal{N}}^{(\nu+1)} = \emptyset$, then terminate. Otherwise, set $\nu := \nu + 1$ and return to Step 1.*

In Step 0, the computational cost for the sort of $\{n_1(k), \dots, n_N(k)\}$ is $O(N \log N)$ for each k. Step 1-1 implies that tone $\overline{n}(k) \in \overline{\mathcal{N}}^{(\nu)}$ is chosen so that $\sigma^{\overline{n}(k)}_k = \min\{\sigma^n_k \mid n \in \overline{\mathcal{N}}^{(\nu)}\}$. In Step 1-2, $\overline{R}'_k$ can be obtained by the water-filling procedure. One is tempted to think Algorithm 3 would always yield a better solution than Algorithm 2. While it often does, numerical results in the next section show that Algorithm 3 sometimes can lead to a worse sum-rate solution than Algorithm 2.

6 Numerical experiment

In this section, we consider a wireless setup and compare the performance of various spectrum management algorithms: (1) the dual decomposition method, (2) the local search algorithms, and (3) the iterative water-filling algorithm (IWFA).

For the dual decomposition method, we choose the initial dual vector $\boldsymbol{\lambda}^{(0)} = (1, \dots, 1)^T$, and consider two different stepsize rules:

Stepsize rule A $\alpha^{(\nu)} = 1/(\nu + 1)$.

Stepsize rule B $\alpha^{(\nu)} := \theta^{(\nu)}(d(\boldsymbol{\lambda}^{(\nu)}) - L^*)/\|g^{(\nu)}\|^2$, where L^* is a known lower bound of the dual function d, and $\theta^{(\nu)}$ is calculated according to the following rule: (i) $\theta^{(0)} = 2$, (ii) $\theta^{(\nu+1)} = \theta^{(\nu)}/2$ if $d(\boldsymbol{\lambda}^{(\nu)}) \geq d(\boldsymbol{\lambda}^{(\nu-10)})$ for $\nu \geq 10$, and (iii) $\theta^{(\nu+1)} = \theta^{(\nu)}$ if $d(\boldsymbol{\lambda}^{(\nu)}) < d(\boldsymbol{\lambda}^{(\nu-10)})$ or $\nu \leq 9$.

In implementing this stepsize rule B, we first calculate the sum-rate by the local search algorithm B, and then use the obtained sum-rate as the lower bound L^*. We stop the algorithm when either $\|\boldsymbol{\lambda}^{(\nu+1)} - \boldsymbol{\lambda}^{(\nu)}\| \leq 10^{-4}$ or $\nu \geq 300$.

For IWFA, we let each user choose an initial power level randomly from the interval $[0, \max_k P_k]$, and terminate the iteration if $\|\mathbf{S}^{(\nu+1)} - \mathbf{S}^{(\nu)}\| \leq 10^{-4}$ or $\nu \geq 300$. As mentioned in Section 1, IWFA maximizes each user's individual rate in a distributed manner by treating other users' signals as Gaussian noise. This can be easily implemented using the well-known water-filling strategy for a single user rate maximization. Since the FDMA concept is not considered in IWFA, the obtained power spectra are not FDMA in general.

In our simulation, we consider a multiuser wireless communication system in a frequency selective environment. We define the channel coefficients as $h^n_{lk} := d^{-1.8}_{lk} g^n_{lk}$ where d_{lk} denotes the physical distance between transmitter l and receiver k, and g^n_{lk} is a complex normalized gaussian random variable with zero mean and unit variance. Then, the crosstalk coefficients and normalized noise power are chosen as $\alpha^n_{lk} := |h^n_{lk}|^2/|h^n_{kk}|^2$ and $\sigma^n_k := N_0/|h^n_{kk}|^2$, where the background noise level is set to $N_0 = -40$ dB. The programs were coded in MATLAB 7 and run on a machine with 3.60GHz CPU and 2GB RAM.

Obtained result

Let there be $N = 12$ tones shared by $K = 4$ users in the system (e.g., the blue tooth setup). Then we randomly generate 4 pairs of transmitters and receivers so that each transmitter k is located in the 2-dimensional unit square and d_{kk} (the distance from transmitter k to receiver k) equals $\Delta > 0$ for all $k \in \mathcal{K}$.[1] Figure 1 shows a simple example, where the solid arrows denote the desired signal path, and all other edges in the graph (not shown) represent interferences.

We let the distance $d_{kk} = \Delta$ vary from 0.02 to 0.2, and generate 1000 test problems for each Δ. As expected, the crosstalk interference becomes stronger when the distance Δ increases. For each test problem, we choose power budget P_k randomly from the interval $[10, 16]$ (dB), and solve the corresponding spectrum management problem by the dual decomposition method with stepsize rules A and B (denoted by dual decomposition method A and B respectively), local search algorithms A and B, and IWFA. The average CPU time among 1000 trials are shown in Figure 2, which shows that the computational costs of the local search algorithms are much lower than other algorithms. Figure 3 shows the average of the obtained sum-rates for each Δ. It can be seen that, for small Δ where the crosstalk coefficients are small, IWFA yields higher sum-rate compared to our FDMA-based methods. This is expected since FDMA is strictly sub-optimal in low interference environment. However, when Δ becomes larger, our FDMA-based methods yield much higher sum-rates than IWFA, confirming the superiority of FDMA solutions under strong crosstalk conditions. Figure 4 plots the ratios of sum-rates obtained by our FDMA-based methods relative to that of the local search algorithm A. As the figure shows, the dual decomposition method B gives the highest average values.

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[1] More precisely, we randomly generate 4 transmitters in the 2-dimensional unit square, and then generate receiver k ($k = 1, \dots, 4$) randomly on the circumference of the circle whose center is transmitter k ($k = 1, \dots, 4$) and radius is Δ.

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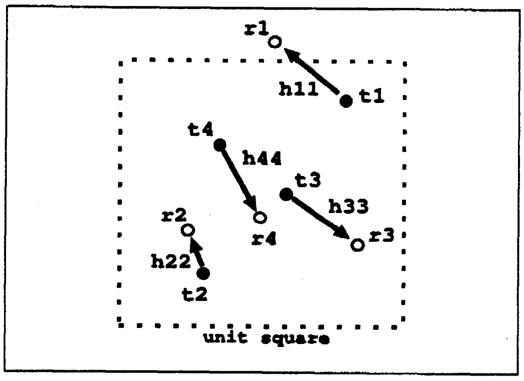

Figure 1: A wireless scenario

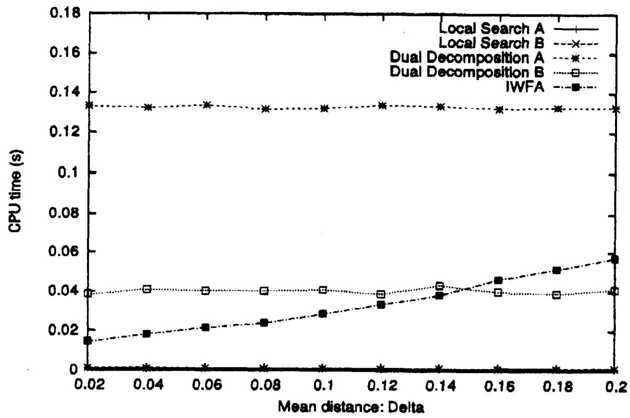

Figure 2: CPU time v.s. distance Δ

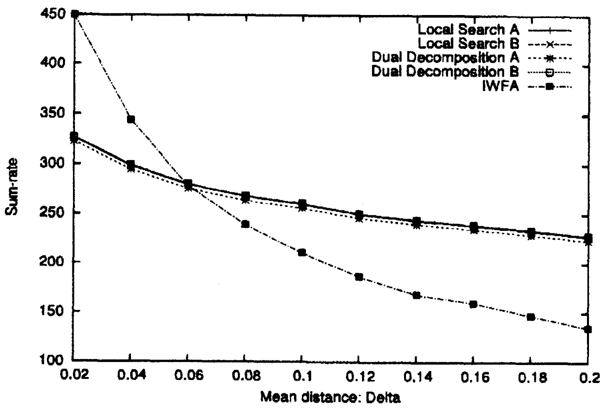

Figure 3: Sum-rate v.s. distance Δ

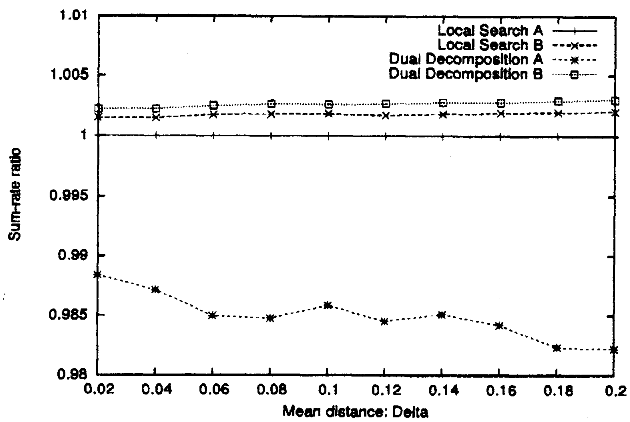

Figure 4: Sum-rate ratios v.s. distance Δ